Classical Annihilation Radiation

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An expression for the total energy $\mathcal{E}_R$ of radiation associated with the classical annihilation of two uniformly moving massless charge distributions which are identical except for sign is obtained. The energy is found to be generated by the processes of bremsstrahlung, pair interaction, and static field cancellation whose contributions give the result $\mathcal{E}_R = 2\,\gamma\,\beta^2\,\mathcal{E}_0$ where $\mathcal{E}_0$ is the charge distribution self-energy. Application of the results to several sample charge distributions supports the conjecture that infinite acceleration of charge does not necessitate infinite radiated energy. A criterion for the finiteness of total emission is established with respect to the self-energies of the charge distributions. For the case of non-zero mass distributions, a formula for the total energy of annihilation is developed, suggesting an annihilation event to be interpretable in terms of the separate annihilation of electromagnetic and inertial mass.

I. Introduction

In this analysis, we wish to examine the problem of the classical annihilation of two massless charge distributions which are identical except for sign. The distributions, each of total charge e, are confined to the planes $z = \pm v\,t$ $(t<0)$, where v is velocity and t is time. We denote by $\mathcal{E}_R$ the total radiation energy released upon annihilation, that is, upon cancellation of charge at collision.

Expressions for both the frequency distribution $\mathcal{E}_R(\omega)$ and total energy $\mathcal{E}_R$ of annihilation radiation are formulated. Application of the expressions obtained to point, ring, and disk-like charge distributions indicates that infinite acceleration of charge does not necessitate infinite radiated energy, but rather, that the total radiation is more directly related to the self-energies of the charge distributions. This observation allows the establishment of a criterion to ascertain the finiteness of total annihilation energy $\mathcal{E}_R$.

One might expect radiation in the annihilation of massless charges simply from the casual cancellation of the static fields[1]. In the course of the analysis, the energy $\mathcal{E}_R{}^S$ associated with this cancellation is found to be one of three effects contributing to the total annihilation energy. The remaining energies, denoted as $\mathcal{E}_R{}^I$ and $\mathcal{E}_R{}^B$, arise respectively from the mutual interaction of the charge distributions prior to annihilation, and the bremsstrahlung radiation

associated with their instantaneous deceleration upon annihilation.

In an attempt to obtain an expression for the total energy of annihilation $\mathcal{E}$ for charge distributions with non-zero mass, the purely electromagnetic energy $\mathcal{E}_R$ associated with the charge annihilation is supplemented by an additional mass annihilation energy $\mathcal{E}_M$. By assuming the observable mass m_0 of each distribution to be the sum of its electromagnetic mass m_e, and inertial mass m_i, an expression for the total energy of annihilation $\mathcal{E}$ is developed. This construction gives results consistent with the relativistically correct annihilation energy $\mathcal{E} = 2\,\gamma\,m_0\,c^2$.

II. Analysis

1. General Formulation

Let two arbitrary planar charge distributions, identical except for sign, approach the origin from
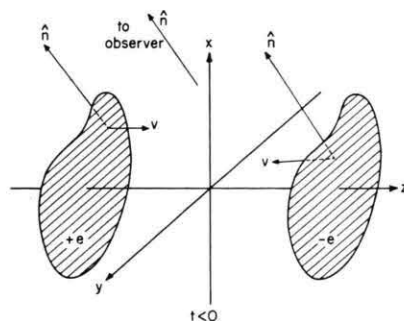

Fig. 1. Two uniformly moving planar charge distributions identical except for sign which are confined to the planes $z = \pm v\,t$, annihilate at $(z, t) = .0$

$(z, t) = \pm \infty$ along the trajectories $z = \pm v\, t\, (t < 0)$ (Figure 1). If $f(\varrho, \varphi)$ denotes the radial and azimuthal charge variation, then each distribution represents a current of magnitude

$$j_z = \begin{cases} (e\, v/2\, \pi)\, f(\varrho, \varphi)\, \delta(z - v\, t) & t < 0, \\ 0 & t \geqq 0, \end{cases} \qquad (1)$$

and a total charge q of

$$q = \frac{e}{2\, \pi} \int\limits_0^\infty \varrho\, \mathrm{d}\varrho \int\limits_0^{2\pi} \mathrm{d}\varphi\, f(\varrho, \varphi) \;.$$

The radiation associated with the annihilation of these massless distributions at $(z, t) = 0$ may be obtained from the relation [2]

$$\frac{\mathrm{d}\mathcal{E}_\mathrm{R}(\omega)}{\mathrm{d}\Omega} = \frac{\omega^2}{4\, \pi^2\, c^3} \left| \int\limits_{-\infty}^{+\infty} \hat{\boldsymbol{n}} \cdot [\hat{\boldsymbol{n}} \cdot \boldsymbol{j}(\boldsymbol{x}, t)] \exp\left\{ i\, \omega \left[t - \frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{x}}{c} \right] \right\} \mathrm{d}^3 x\, \mathrm{d}t \right|^2 \qquad (2)$$

which gives the energy $\mathcal{E}_\mathrm{R}$ radiated per frequency interval $\mathrm{d}\omega$ per solid angle $\mathrm{d}\Omega$ in the direction $\hat{n}$ due to the current $\boldsymbol{j}$. Noting the relations

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{x} = \varrho \sin\theta \cos(\varphi - \varphi') + z \cos\theta\,, \qquad |\hat{\boldsymbol{n}} \cdot (\hat{\boldsymbol{n}} \cdot \boldsymbol{j})| = j \sin\theta\,,$$

Eq. (2) becomes

$$\frac{\mathrm{d}\mathcal{E}_\mathrm{R}(\omega)}{\mathrm{d}\Omega} = \left(\frac{\omega\, e\, v \sin\theta}{2\, \pi\, c^2} \right)^2 c \times$$

$$\times \left| (\int\limits_{-\infty}^0 \exp\{i\, \omega(1 - \beta \cos\theta)\, t\}\, \mathrm{d}t + \int\limits_{-\infty}^0 \exp\{i\, \omega(1 + \beta \cos\theta)\, t\}\, \mathrm{d}t) \int\limits_0^\infty \varrho'\, \mathrm{d}\varrho' \int\limits_0^{2\pi} f(\varrho', \varphi') \right.$$

$$\left. \times \exp\left\{ -\frac{i\, \omega\, \varrho'}{c}\, \Lambda(\theta, \varphi - \varphi') \right\} \frac{\mathrm{d}\varphi'}{2\, \pi} \right|^2\,, \qquad (3)$$

$$\Lambda(\theta, \varphi - \varphi') \equiv \sin\theta \cos(\varphi - \varphi')$$

wherein we have made a coherent superposition of the contributions from each plane. The time integration in (3) is easily performed yielding

$$\frac{\mathrm{d}\mathcal{E}_\mathrm{R}(\omega)}{\mathrm{d}\Omega} = \left(\frac{e\, v}{\pi\, c^2} \right)^2 c\, \frac{\sin^2\theta}{(1 - \beta^2 \cos^2\theta)^2} \left| \int\limits_0^\infty \varrho'\, \mathrm{d}\varrho' \int\limits_0^{2\pi} \frac{\mathrm{d}\varphi'}{2\, \pi}\, f(\varrho', \varphi') \exp\left\{ -\frac{i\, \omega\, \varrho'}{c}\, \Lambda(\theta, \varphi - \varphi') \right\} \right|^2\,. \qquad (4)$$

The frequency spectrum $\mathcal{E}_\mathrm{R}(\omega)$ due to the $t = 0$ annihilation is for a given $f(\varrho, \varphi)$ obtained by an integration of (4) over $4\, \pi$ solid angle. The total radiated energy $\mathcal{E}_\mathrm{R}$, however, requires two integrations, as follows.

First, we integrate over ω, keeping θ and φ fixed, by introducing the change of variable $\omega = \varkappa\, c/\sin\theta$. The integration over θ may then be carried out with the result

$$\int\limits_0^\pi \frac{\sin^2\theta\, \mathrm{d}\theta}{(1 - \beta^2 \cos^2\theta)^2} = \frac{\pi}{2\, \sqrt{1 - \beta^2}} = \frac{\pi}{2}\, \gamma$$

enabling (4) to be written

$$\frac{\mathrm{d}\mathcal{E}_\mathrm{R}(\varkappa)}{\mathrm{d}\varphi} = \frac{e^2\, \gamma\, \beta^2}{2\, \pi} \left| \int\limits_0^\infty \varrho'\, \mathrm{d}\varrho' \int\limits_0^{2\pi} \frac{\mathrm{d}\varphi'}{2\, \pi}\, f(\varrho', \varphi') \exp\{ -i\, \varkappa\, \varrho' \cos(\varphi - \varphi') \} \right|^2\,. \qquad (5)$$

We now rearrange (5) in such a manner as to allow the explicit integration over φ.

By expressing the exponential in terms of its real and imaginary parts, the absolute value signs in (5) may be removed to yield

$$\frac{d\mathcal{E}_R(\varkappa)}{d\varphi} = \frac{e^2 \beta^2 \gamma}{2\pi} \left[\left(\frac{1}{2\pi} \int_0^\infty \varrho' \, d\varrho' \int_0^{2\pi} f(\varrho', \varphi') \cos[\varkappa \varrho' \cos(\varphi - \varphi')] \, d\varphi' \right)^2 \right.$$

$$\left. + \left(\frac{1}{2\pi} \int_0^\infty \varrho' \, d\varrho' \int_0^{2\pi} f(\varrho', \varphi') \sin[\varkappa \varrho' \cos(\varphi - \varphi')] \, d\varphi' \right)^2 \right]. \tag{6}$$

Recalling the identities [3]

$$\cos(x \cos \theta) = 2 \sum_{n=0}^\infty{}' (-1)^n J_n(x) \cos 2 n \theta,$$

$$\sin(x \cos \theta) = 2 \sum_{n=0}^\infty (-1)^n J_{2n+1}(x) \cos(2n+1)\theta$$

where the prime weights the $n = 0$ term with a factor $1/2$, (6) becomes

$$\frac{d\mathcal{E}_R(\varkappa)}{d\varphi} = \frac{2}{\pi} e^2 \gamma \beta^2 \left[\left(\sum_{n=0}^\infty{}' A_{2n} \cos 2n\theta + \sum_{n=0}^\infty{}' B_{2n} \sin 2n\theta \right)^2 \right.$$

$$\left. + \left(\sum_{n=0}^\infty A_{2n+1} \cos(2n+1)\theta + \sum_{n=0}^\infty B_{2n+1} \sin(2n+1)\theta \right)^2 \right] \tag{7}$$

wherein

$$A_n \equiv \frac{1}{2\pi} \int_0^\infty \varrho' \, d\varrho' \, J_n(\varkappa \varrho') f(\varrho', \varphi') \int_0^{2\pi} \cos 2n\varphi' \, d\varphi',$$

$$B_n \equiv \frac{1}{2\pi} \int_0^\infty \varrho' \, d\varrho' \, J_n(\varkappa \varrho') f(\varrho', \varphi') \int_0^{2\pi} \sin 2n\varphi' \, d\varphi'. \tag{8}$$

The ϱ' integration in (8) is recognized to be the nth order Hankel transform of the charge distribution, which we denote as $\tilde{f}_n(\varkappa, \varphi)$. Thus,

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_n(\varkappa, \varphi) \cos 2n\varphi \, d\varphi,$$

$$B_n = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_n(\varkappa, \varphi) \sin 2n\varphi \, d\varphi. \tag{9}$$

Returning to (7), it is evident that upon integration over φ, all squared terms contribute $\frac{1}{2}(2\pi)$, whereas all cross terms vanish. Consequently,

$$\mathcal{E}_R(\varkappa) = 2 e^2 \gamma \beta^2 \sum_{n=0}^\infty{}' (A_n^2 + B_n^2)$$

$$= 2 e^2 \gamma \beta^2 \sum_{n=0}^\infty{}' |A_n + i B_n|^2. \tag{10}$$

Finally, employing (9), the annihilation energy $\mathcal{E}_R$ is found to be

$$\mathcal{E}_R = 2 e^2 \gamma \beta^2 \sum_{n=0}^\infty{}' \int_0^\infty \left| \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_n(\varkappa, \varphi) e^{in\varphi} \, d\varphi \right|^2 d\varkappa. \tag{11}$$

In the case of a planar charge distribution which enjoys azimuthal symmetry, this assumes the much simpler form

$$\mathcal{E}_R = e^2 \gamma \beta^2 \int_0^\infty \tilde{f}^2(\varkappa) \, d\varkappa$$

where $\tilde{f}(\varkappa)$ is the zero-order Hankel transform of the radial charge distribution.

In order to write (11) in a more concise form, we introduce the finite Fourier transform $\tilde{f}(n)$ given by

$$\tilde{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) e^{in\varphi} \, d\varphi,$$

$$f(\varphi) = \sum_{n=-\infty}^{+\infty} \tilde{f}(n) e^{-in\varphi}.$$

Table 1

Planar charge distribution	Charge Transform	Annihilation Energy
Arbitrary planar charge $f(\varrho, \varphi)$	$\tilde{\tilde{f}}(\varkappa, n) = \frac{1}{2\pi} \int\limits_0^{2\pi} [\int\limits_0^\infty f(\varrho, \varphi) J_n(\varkappa \varrho) \varrho \, d\varrho] \, e^{in\varphi} \, d\varphi$	$\mathcal{E}_R = 2 \, e^2 \, \gamma \, \beta^2 \sum\limits_{n=0}^\infty {}'\int\limits_0^\infty |f(\varkappa, n)|^2 \, d\varkappa$
Arbitrary planar charge with azimuthal symmetry $f(\varrho)$	$\tilde{f}(\varkappa) = \int\limits_0^\infty f(\varrho) J_0(\varkappa \varrho) \varrho \, d\varrho$	$\mathcal{E}_R = e^2 \, \gamma \, \beta^2 \int\limits_0^\infty \tilde{f}^2(\varkappa) \, d\varkappa$
Point charges $f(\varrho) = \delta(\varrho)$	$\tilde{f}(\varkappa) = 1$	$\mathcal{E}_R = \infty$
Charged rings $f(\varrho) = \delta(\varrho - r_0)$	$\tilde{f}(\varkappa) = J_0(\varkappa r_0)$	$\mathcal{E}_R = \infty$
Charged disks $f(\varrho) = a \begin{cases} 1 & \varrho \leq r_0 \\ 0 & \varrho > r_0 \end{cases}$	$\tilde{f}(\varkappa) = \frac{a}{\varkappa} J_1(\varkappa r_0)$	$\mathcal{E}_R = e^2 \, \gamma \, \beta^2 \left(\frac{4}{3\sqrt{\pi}} \, a^2 \, r_0^3 \right)$

There results

$$\mathcal{E}_R = 2 \, e^2 \, \gamma \, \beta^2 \sum_{n=0}^\infty {}' \int\limits_0^\infty |\tilde{\tilde{f}}(\varkappa, n)|^2 \, d\varkappa \; .$$

Three applications of (11) have been considered. These include: a) Two point charges, b) Two charged rings of radius r_0, and c) Two charged disks of radius r_0. Results for these examples are tabulated in Table 1.

In obtaining (11), each charge distribution was forced to experience an infinite acceleration at $(z, t) = 0$ in its instantaneous transition from a uniform velocity v to rest. Such a singular acceleration is not sufficient, however, to generate an infinity of radiated energy. This is apparent from both the charged disk example contained in Table 1 and an investigation of the classical expression

$$\mathcal{E}_R = P_R \, \varDelta t \propto a^2 \, \varDelta t \tag{12}$$

where the radiated energy $\mathcal{E}_R$ is given in terms of the radiated power P_R, which is proportional to the square of the acceleration a, and the time $\varDelta t$ during which the acceleration occurs. Since $a \to \infty$ and $\varDelta t \to 0$, (12) is indeterminant. As it turns out, the integral in (11) over the square of the transformed charge distribution is found to be proportional to the self-energy of the charge distribution $\mathcal{E}_0$. This correspondence allows an unambiguous criterion for finite radiated energy. To illustrate this correspondence between self and radiated energies, we now calculate the electrostatic self-energy associated with a general planar charge distribution.

2. Self-Energy

In the rest frame of the moving distribution, the self-energy $\mathcal{E}_0$ is given in terms of the charge density $\varrho(\boldsymbol{x})$ and potential $\Phi(\boldsymbol{x})$ in the form

$$\mathcal{E}_0 = \tfrac{1}{2} \int \Phi(\boldsymbol{x}) \varrho(\boldsymbol{x}) \, d\boldsymbol{x} \; . \tag{13}$$

In general, the static potential $\Phi(x)$ is given by

$$\Phi(\boldsymbol{x}) = \int \varrho(\boldsymbol{x}') \, d\boldsymbol{x}' / |\boldsymbol{x} - \boldsymbol{x}'| \tag{14}$$

where, in the present problem

$$\varrho(\boldsymbol{x}) = (e/2\pi) f(\varrho, \varphi) \delta(z) \; .$$

Returning these expressions to (13) and making use of the representation [4]

$$\frac{1}{|x - x'|} = \sum_{m=-\infty}^{+\infty} \int\limits_0^\infty J_m(\varkappa \varrho) J_m(\varkappa \varrho')$$

$$\cdot \exp\{-i \, m(\varphi - \varphi')\} \exp\{-\varkappa |z - z'|\} \, d\varkappa \tag{15}$$

there results

$$\mathcal{E}_0 = \tfrac{1}{2} e^2 \sum_{n=-\infty}^{+\infty} \int\limits_0^\infty \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \tilde{f}_n(\varkappa, \varphi') \, e^{im\varphi'} \, d\varphi' \right)$$

$$\cdot \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \tilde{f}_n(\varkappa, \varphi) \, e^{-im\varphi} \, d\varphi \right) d\varkappa$$

which is easily rearranged as

$$\mathcal{E}_0 = e^2 \sum_{n=0}^\infty \int\limits_0^\infty \left| \frac{1}{2\pi} \int\limits_0^{2\pi} \tilde{f}_n(\varkappa, \varphi) \, e^{in\varphi} \, d\varphi \right|^2 d\varkappa \; . \tag{16}$$

3. Annihilation Energy

Equation (16) represents the electrostatic self-energy available to each of the planar charge distributions in an annihilation collision. A comparison with (11) indicates

$$\mathcal{E}_{\mathrm{R}} = 2\,\gamma\,\beta^2\,\mathcal{E}_0 . \tag{17}$$

That is, the total energy lost to radiation by each charge distribution is directly proportional to its self-energy. As noted above, this result provides an unambiguous criterion by which to establish the finiteness of the total emission: all annihilating distributions whose self-energies are finite will give rise to a finite radiation energy.

We will now show that the annihilation radiation $\mathcal{E}_{\mathrm{R}}$ as given by (17) may be attributed to three distinct processes. First, each distribution, as a result of its instantaneous deceleration at $(z, t) = 0$, gives rise to bremsstrahlung radiation energy $\mathcal{E}_{\mathrm{R}}^{\mathrm{B}}$. This contribution, due to the deceleration of a single distribution, is easily calculated and given by (2),

$$\frac{\mathrm{d}\mathcal{E}_{\mathrm{R}}^{\mathrm{B}}(\omega)}{\mathrm{d}\Omega} = \left(\frac{\omega\,e\,v\,\sin\theta}{2\,\pi\,c^2}\right)^2 c$$

$$\cdot \left| \int\limits_{-\infty}^{0} \exp\{i\,\omega\,(1\pm\beta\cos\theta)\,t\}\,\mathrm{d}t \int\limits_{0}^{\infty} \varrho'\,\mathrm{d}\varrho' \int\limits_{0}^{2\pi} f(\varrho',\varphi')\exp\left\{-i\,\frac{\omega}{c}\,\varrho\,\varLambda\,(\theta,\varphi-\varphi')\right\} \frac{\mathrm{d}\varphi'}{2\,\pi} \right|^2 .$$

The only difference between this expression and that of Eq. (4) is the θ integration, which in this case is

$$\int\limits_{0}^{2\pi} \frac{\sin^2\theta\,\mathrm{d}\theta}{(1-\beta\cos\theta)^2} = \frac{\pi}{\beta^2}\,(\gamma-1) .$$

Thus,
$$\mathcal{E}_{\mathrm{R}}^{\mathrm{B}} = (\gamma-1)\,\mathcal{E}_0 .$$

In stopping, then, each charge radiates what might be termed its "electromagnetic kinetic energy".

Second, an interaction energy $\mathcal{E}_{\mathrm{R}}^{\mathrm{I}}$ between the charges also appears upon annihilation. This energy is equal to the total work done by the two charges and found to be

$$\mathcal{E}_{\mathrm{R}}^{\mathrm{I}} = -\,2\,\mathcal{E}_0/\gamma \tag{18}$$

(see Appendix).

Adding these two energies, we obtain

$$2\,\mathcal{E}_{\mathrm{R}}^{\mathrm{B}} + \mathcal{E}_{\mathrm{R}}^{\mathrm{I}} = 2\,\mathcal{E}_0\,(\gamma\,\beta^2 - 1) = \mathcal{E}_{\mathrm{R}} - 2\,\mathcal{E}_0$$

or equivalently,

$$2\,(\mathcal{E}_{\mathrm{R}}^{\mathrm{B}} + \mathcal{E}_{\mathrm{R}}^{\mathrm{S}}) + \mathcal{E}_{\mathrm{R}}^{\mathrm{I}} = \mathcal{E}_{\mathrm{R}} . \tag{19}$$

Here, we have made the identification

$$\mathcal{E}_{\mathrm{R}}^{\mathrm{S}} = \mathcal{E}_0 .$$

The term $\mathcal{E}_{\mathrm{R}}^{\mathrm{S}}$ is interpreted as the energy associated with the cancellation of the static fields. Thus, a sequence of events contributing to the annihilation of distributions is inferred in which the "particles" release energies of bremsstrahlung $\mathcal{E}_{\mathrm{R}}^{\mathrm{B}}$, interaction $\mathcal{E}_{\mathrm{R}}^{\mathrm{I}}$, and static field cancellation $\mathcal{E}_{\mathrm{R}}^{\mathrm{S}}$.

The presence of both $\mathcal{E}_{\mathrm{R}}^{\mathrm{S}}$ and $\mathcal{E}_{\mathrm{R}}^{\mathrm{I}}$ demonstrates that the annihilation energy of two opposite charges is not merely the separate bremsstrahlung of the charges, that is

$$\mathcal{E}_{\mathrm{R}} \neq 2\,\mathcal{E}_{\mathrm{R}}^{\mathrm{B}} .$$

This inequality is also evident from (4) whose two squared terms, which do represent the separate bremsstrahlung contributions $2\,\mathcal{E}_{\mathrm{R}}^{\mathrm{B}}$, are supplemented by a cross term or "interference" term. This interference term contains both the static and interaction energies. These various contributions are depicted schematically in the space-time diagram
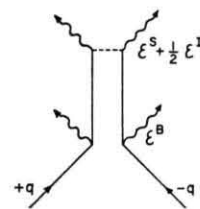

Fig. 2. Graphical Illustration depicting processes in the classical annihilation of two charge distributions. The vertical time axis is expanded to distinguish the bremsstrahlung events from that of the release of static and annihilation energy.

shown in Figure 2. The ordering of events suggests a process in which the particles first release bremsstrahlung energy in stopping and then static and interaction energies in annihilation [5].

4. Annihilation Including Mass

In the case that the distributions contain mass as well as charge, the energy liberated in the annihilation event may be considered to arise from two

sources. First, "charge" annihilation which contributes the purely electromagnetic energy $\mathcal{E}_R$. This radiation may be attributed to the annihilation of "electromagnetic mass" m_e, a quantity we take to represent the inertial property of the fields. The energy liberated may then be written in the form

$$\mathcal{E}_R = 2\,\gamma\,\beta^2\,\mathcal{E}_0 \equiv 2\,\gamma\,m_e\,c^2 . \qquad (20)$$

Second, the "mass" annihilation, whose energy $\mathcal{E}_M$ must be added to (20). For charge distributions of inertial mass m_i this energy is simply

$$\mathcal{E}_M = 2\,\gamma\,m_i\,c^2 .$$

By now assuming [6] that the observable rest mass m_0 of an annihilating charge distribution is the sum of both the electromagnetic mass m_e and inertial mass m_i,

$$m_0 = m_e + m_i$$

we find for the total energy $\mathcal{E}$ of annihilation

$$\mathcal{E} = 2\,\gamma\,m_e\,c^2 + 2\,\gamma\,m_i\,c^2 = 2\,\gamma\,m_0\,c^2 .$$

Thus, the total annihilation event may be thought of as the sum annihilations in both electromagnetic and inertial mass. This latter equation represents the correct relativistic form for the total annihilation energy.

Acknowledgements

Appendix

The Interaction Energy

The interaction energy $\mathcal{E}_R^I$ is equal to the work done by the two charge distributions depicted in Fig. 1 in moving uniformly from $z = \pm\infty$ to the origin. We will subsequently denote by A the distribution confined to $z < 0$, and by B that in the space $z > 0$. By symmetry, the work W performed by each distribution is the same, so that a calculation of W for A is sufficient. Clearly, $\mathcal{E}_R^I = 2\,W$.

According to Poynting's theorem, the work done by A against the fields of B is given by

$$W = -\int_{-\infty}^{0} \mathrm{d}t \int \mathbf{E} \cdot \mathbf{j}\, \mathrm{d}\mathbf{x} \qquad (21)$$

where

$$\mathbf{j} = \hat{k}\,(e\,v/2\,\pi)\,f(\varrho)\,\delta(z - v\,t) .$$

is the current associated with $A\,(t<0)$, and $\mathbf{E}$ the electric field generated by B. For the purpose of the present calculation, we assume the charge distributions to be azimuthally symmetric, i. e., $f(\varrho, \varphi) \to f(\varrho)$. It is not difficult to show, however, that the results expressed by (18) are also valid for arbitrary $f(\varrho, \varphi)$.

The field E_z due to B is most easily determined by summing contributions from charged rings. For a ring of charge $-\varDelta e$ and radius ϱ' which is located at the origin with its center on the z-axis and lying in the xy-plane, the potential Φ for $z < 0$ is given by [7]

$$\Phi(\varrho, z) = -\varDelta e \int_0^\infty J_0(\varkappa\,\varrho)\,J_0(\varkappa\,\varrho')\,e^{\varkappa z}\,\mathrm{d}\varkappa$$

with which we associate a static electric field

$$E_z^0 = -\partial\Phi/\partial z = \varDelta e \int_0^\infty J_0(\varkappa\,\varrho)\,J_0(\varkappa\,\varrho')\,\varkappa\,e^{\varkappa z}\,\mathrm{d}\varkappa . \quad (22)$$

If now the charged ring is allowed to move uniformly with velocity v, its field E_z, for $t < 0$, may be obtained by a Lorentz transformation [8] of (22), yielding

$$E_z = \varDelta e \int_0^\infty J_0(\varkappa\,\varrho)\,J_0(\varkappa\,\varrho')\,\varkappa\,e^{\varkappa\gamma(z + v t)} .$$

Since the charge distribution B may be considered to be composed of an infinite number of such moving rings each of infinitesimal charge $-\varDelta e = -e\,2\,\pi\,\varrho'\,\mathrm{d}\varrho'\,f(\varrho')$, the total field due to B for $z < 0, t < 0$, is

$$E_z = 2\,\pi\,e \int_0^\infty f(\varrho')\,\varrho'\,\mathrm{d}\varrho' \int_0^\infty J_0(\varkappa\,\varrho)\,J_0(\pi\,\varrho')\,\varkappa\,e^{\varkappa\gamma(z + v t)}$$

which may be rewritten in the form

$$E_z = 2\,\pi\,e \int_0^\infty \tilde{f}(\varkappa)\,J_0(\varkappa\,\varrho)\,\varkappa\,e^{\varkappa\gamma(z + v t)}\,\mathrm{d}\varkappa .$$

Here, $\tilde{f}(\varkappa)$ is the zero-order Hankel transform of the radial charge distribution $f(\varrho)$. Returning both this expression for E_z and that of the current $\mathbf{j}$ to (21) yields

$$W = -e^2\,v \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}z\,\delta(z - v\,t) \int_0^\infty \tilde{f}^2(\varkappa)\,\varkappa\,e^{\varkappa\gamma(z + v t)} .$$

Introducing the change of variable

$$\begin{aligned} x &= z - \xi \\ y &= z + \xi \end{aligned} \qquad \xi = v\,t$$

which carries a Jacobian of 1/2, there results

$$W = -\frac{1}{2\gamma} \int_0^\infty \tilde{f}^2(\varkappa)\, d\varkappa.$$

Comparison with (16) indicates $W = -\mathcal{E}_0/\gamma$, and consequently

$$\mathcal{E}_R{}^I = -2\,\mathcal{E}_0/\gamma.$$

[1] A pictoral representation of the Coulomb field relevant to this event is given by E. Purcell in Electricity and Magnetism, Berkeley Physics course, Vol. II, McGraw-Hill Book Co., New York 1965, p. 165.

[2] J. D. Jackson, Classical Electrodynamics, John Wiley & Sons, New York 1968, p. 481.

[3] G. Arfken, Mathematical Methods for Physicists, Academic Press, New York 1968, p. 385 pb. 11. 1. 28.

[4] See reference [2], p. 96.

[5] Note that the inferred composite structure of this event is somewhat similar to the succession of events connected with the well known quantum electrodynamic second order (isolated) electron-positron pair annihilation. For further discussion see S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Harper and Row, New York 1962.

[6] F. Rohrlich, Classical Charged Particles, Addison-Wesley Co. Inc. Mass., 1965. The decomposition of mass into electromagnetic and inertial parts is discussed on page 10.

[7] This expression is derived by employing in (14) both the representation (15), and the charge density $\varrho(x)$ appropriate to a ring.

[8] This technique of obtaining dynamic from static fields by Lorentz transformation is also demonstrated by R. L. Liboff, J. Math. Physics 11, 1295 [1970]; 13, 1828 [1972].